Exc1

$$A = \begin{bmatrix} -1 & 101 & & & & \\ -99 & -2 & 101 & & & \\ & -99 & -2 & 101 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -99 & -2 & 101 \\ & & & & -99 & -2 \end{bmatrix}$$

a. show A irreducible

graph of A: (1) ↔ (2) ↔ (3) ↔ ⋯ ↔ (99) ↔ (100)

graph strongly connected, since all nodes are connected to each other, path from node i to j an vice verse for all i and j

⇒ A irreducible

b. $A = \frac{A^T + A}{2} + \frac{A^T - A}{2} = \begin{bmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix} + \begin{bmatrix} 0 & 100 & & & \\ -100 & 0 & 100 & & \\ & \ddots & \ddots & \ddots & \\ & & -100 & 0 & 100 \\ & & & -100 & 0 \end{bmatrix}$

A_1 symm. $\quad$ A_2 skew symm

0.5

- A_1 symm ⇒ $\lambda \in \sigma(A_1)$ real
  - Gerschgorin on A_1:
    - $C_1 = \{z \in \mathbb{C} \mid |z+2| \le 2\}$
    - first row $C_2 = \{z \in \mathbb{C} \mid |z+1| \le 1\}$
    - last row $C_3 = \{z \in \mathbb{C} \mid |z+2| \le 1\}$

    $\Rightarrow \sigma(A_1) \subset \bigcup_{i=1}^{3} C_i$
  - $\sigma(A_1)$ real ⇒ $\sigma(A_1) \subset [-4, 0] = [-3,-1] \cup [-4,0] \cup [-2,0]$

0.8

A_1 irreducible ⇒ boundary included only if boundary on every circle

⇒ $-4, 0$ not included

⇒ $\sigma(A_1) \subset (-4, 0)$

A_1 symmetric, hence looking at A_1^T does not give additional information

- A_2 skew symm ⇒ $\lambda \in \sigma(A_2)$ pure imaginary

  A_2 irreducible ⇒ boundaries Gerschgorin circles only included if on every circle

  $C_1 = \{ |z| \le 100 \}$

  $C_2 = \{ |z| \le 200 \}$

  $\Rightarrow \sigma(A_2) \subset \bigcup_{i=1}^{2} C_i = [-200i, 200i]$ pure imaginary

0.7

boundary not included : $\sigma(A_2) \subset (-200i, 200i)$

$C_1 \subset C_2$

c) Bendixson : $F(A) \subset F\left(\frac{A+A^T}{2}\right) + F\left(\frac{A-A^T}{2}\right)$

0.4 we have - $\sigma(A) \subset F(A)$

- if A normal, i.e. $AA^T = A^T A \Rightarrow F(A)$ convex hull of $\sigma(A)$

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* $\frac{A+A^T}{2}$ normal $\Rightarrow F\left(\frac{A+A^T}{2}\right)$ = convex hull of $\sigma\left(\frac{A+A^T}{2}\right)$ $= (-4, 0)$

0.4 * $\frac{A-A^T}{2}$ normal $\Rightarrow F\left(\frac{A-A^T}{2}\right)$ = convex hull of $\sigma\left(\frac{A-A^T}{2}\right)$ $= (-200i, 200i)$

0.2 $\Rightarrow \sigma(A) \subset \{x + iy \mid x \in (-4, 0), y \in (-200i, 200i)\}$

Exc 2 A real symmetric, x, with $\|x\|_2 = 1$

(x, θ) Ritz pair obtained from Lanczos

show $\min_{\lambda \in \sigma(A)} |\lambda - \theta| \leq \|Ax - \theta x\|_2$

0.4 • A real symm $\Rightarrow \exists Q$ orthogonal s.t. $Q^T A Q = D$

$D_i = \lambda_i \quad \lambda_i \in \sigma(A)$

$\Rightarrow \|Ax - \theta x\|_2 = \|Q^T(Ax - \theta x)\|_2$ ($\uparrow$ Q orthogonal)

0.8 $= \|Q^T(AQy - \theta Qy)\|_2 = \|Dy - \theta y\|_2$

($\uparrow$ $x = Qy$, $(\exists y$ s.t. $x = Qy)$)

$= \sqrt{\sum_{i=1}^{n} (\lambda_i - \theta)^2 y_i^2}$

$\geq \min_{\lambda \in \sigma(A)} |\lambda - \theta| \sqrt{\sum |y_i|^2}$

0.8 $= \min_{\lambda \in \sigma(A)} |\lambda - \theta| \, \|y\|_2$

$= \min_{\lambda \in \sigma(A)} |\lambda - \theta| \, \|x\|_2 = \min_{\lambda \in \sigma(A)} |\lambda - \theta|$

($\uparrow$ $\|y\|_2 = \|Q^T x\|_2 = \|x\|_2$) ($\uparrow$ $\|x\|_2 = 1$)

Exc 3

a) QR - method

$A^{(0)} = A$

for $i = 0, 1, 2, 3, \ldots$

0.5 $Q^{(i)} R^{(i)} = A^{(i)}$

$A^{(i+1)} = R^{(i)} Q^{(i)}$

end

for general matrices this converges to the real Schur form of the matrix A, which may have

0.5 some 2×2 blocks on the diagonal if A has complex eigenvalues

b) reduction of off-diagonal elements

(2,1) coefficient : decrease factor $\frac{\lambda_2}{\lambda_1} \approx \frac{3}{4.7} \approx 0.6$

0.5 (3,2) " " " $\frac{\lambda_3}{\lambda_2} \approx \frac{1.3}{3} \approx 0.43$

this is indeed the case

$0.2979 \times 0.6 \approx 0.1924$

0.5 $0.0274 \times 0.43 \approx 0.115$

c) apply QR - step with shift to the middle matrix by which factor will the (3,2) element decrease approximately?

apply shift μ ⇒ ~~[illegible]~~ eigenvalues become $\lambda_3 - \mu,\ \lambda_2 - \mu,\ \lambda_1 - \mu$

⇒ element (3,2) will decrease by factor $\frac{\lambda_3 - \mu}{\lambda_2 - \mu}$

[if shift = element 3,3 of middle matrix 1.2684

not required? factor for element (3,2) $\frac{1.2679 - 1.2684}{3.0 - 1.2684} \approx \ldots$